Solvable problems

Simplest problem is the "free particle", i.e., no potential.

$$H_{op} = -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

(In one dimension $H_{op} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$).

Solution

$\psi(x, y, z) = \varphi(x)\chi(y)\eta(z);$

$\varphi(x) = A\sin k_x x + B\cos k_x x = Ce^{ik_x x} + De^{-ik_x x}$

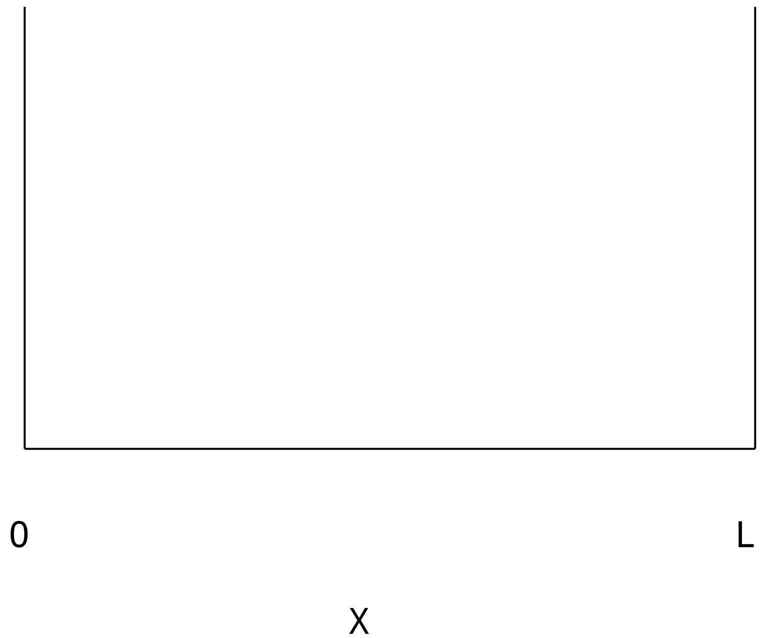
and similar expressions for y and z. **k** is the <u>wavevector</u>

k = **x**k_x+**y**k_y+ **z**k_z The energy is $E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$

This is an important solution for scattering problems; however, it also show up in the simplest **bound state** problem.

A huge class of problems are "bound state" ones, where the system is confined by a potential to a finite region of space. Solutions require proper bound state "boundary conditions" Schematically, this means that the wave function must be zero (or tending to zero) for values of coordinates beyond the "region of confinement".

I. *Particle-in-a-box* (potential with infinite walls)



The boundary conditions are simple; the wave function must vanish at x = 0 and also at x = L. Inside the particle is “free”, i.e., there is no potential. The solutions satisfying this conditions at x = 0 is **of the form** $\varphi(x) = A \sin k_x x$**. Why does the cosine term vanish? At x = L**

$\varphi(L) = A \sin k_x L = 0 \quad k_x = \frac{n\pi}{L}, n = 1,2,3....$ **Thus, k_x can only take <u>quantized</u> values and** $E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$ **is also <u>quantized</u>.**

What about A? Require the wave function to be normalized. That is

$$1 = \int_{-\infty}^{\infty} dx \varphi^*(x)\varphi(x) = |A|^2 \int_0^L dx \sin^2 \frac{n\pi x}{L}. \quad A = \sqrt{\frac{2}{L}}$$